



COUNTING WITH BINOMIALS

FUN, THINKING, FRIENDSHIP

Feel free to talk to one another about the maths and about things that interest you.

1. What do we know?

Pascal's Triangle

- (1) By recursion with numbers and algebra. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
- (2) Relationship with binomial expansion.
- (3) Some properties (web page): symmetry, row sums, primes, hockey stick, Fibonacci, fractals, Sierpinski's Carpet.
- (4) Algebraic definition of each term.
- (5) Side items: factorials, evaluation of $\binom{n}{k}$ (and online calculator), combinations, Triangle numbers: $\sum_{i=1}^n i$ (and ways to get it).
- (6) For practise, $t_4 =$
4 3 2 1 0
3 2 1 0
2 1 0
1 0
0

Catalan Numbers

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

- (1) Web: Explore how Catalan Numbers arise.
- (2) Find C_1, C_2, C_3, C_4, C_5 for a couple of models.
- (3) Find $\sum_{i=1}^n C_i$ for $n = 1, 2, 3, 4, 5$.

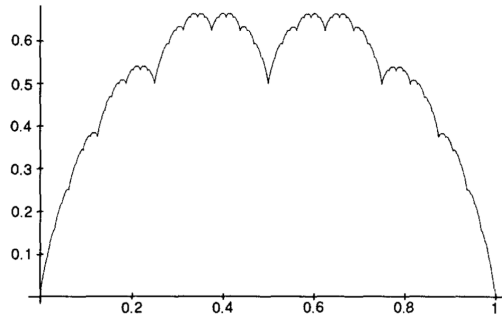
Resources: Pascal's Triangle, your head, your calculator, online list of Catalan numbers, or the online binomial calculator.

2. Some motivation: Extreme values, continuous v. Discrete

Given the discrete KK-function graph (discrete) v. Takagi function (continuous, nowhere differentiable).
Want to do the analogue of calculus but for the discrete function e.g. find maximum.

The continuous Takagi Function (1903). Takagi was a Japanese mathematician.. Symmetric, nowhere differentiable, self-similar.

The following picture is the graph of the Takagi function:



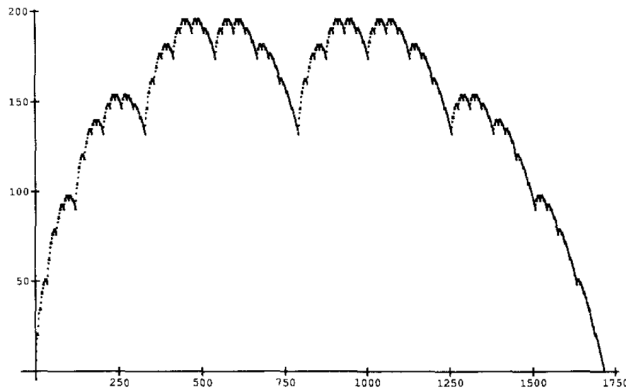
Source: Frankl, Matsumoto, Ruzsa, Tokushige, *Minimum Shadows in Uniform Hypergraphs and a Generalization of the Takagi Function (1995)*

One of the main roles of calculus is to find the critical or extreme values (such as minimums and maximums) of functions. Calculus works very well for 'nice' continuous functions. 'Nice' needs a technical definition and this is part of the role of Mathematical Analysis. The Takagi function is not 'nice' in the sense that it is continuous but nowhere differentiable.

Calculus is of little use for most Discrete functions (sets of single points). However we want to be able to consider discrete functions to obtain corresponding critical or extreme values. We will consider such a problem with a particular discrete function.

The related discrete KK-Function: $n = 13, k = 7$.

There are $\binom{13}{7} + 1 = 1716 + 1$ points in its domain. It is not symmetric (density and symmetry).

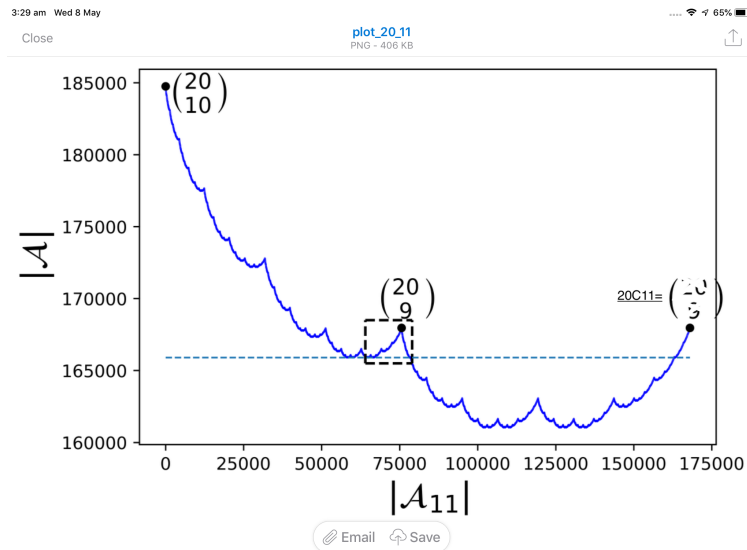


Some standard questions are: What is its maximum value? How often does the maximum get achieved? Local minimums/maximums? What values are/are not achieved by the function? (the range) What is the most common (modal) value? When does it increase/decrease (positive/negative slope of the corresponding linear interpolation function)?

What's the minimum/maximum answers for: a polynomial of degree n ? the exponential function? the sin function?

The related $(k, k - 1)$ FSFA size function on $[20]$ defined pointwise for $k = 11$. There are $\binom{20}{11} + 1 = 167960 + 1$ points.

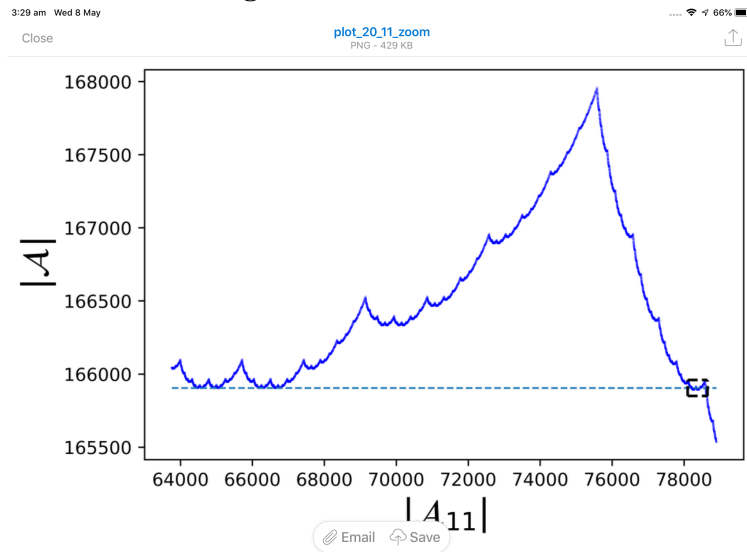
Is there an underlying geometry? What is it?



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There are $2^9 = 256$ occurrences of the minimum value $\binom{20}{11} - \sum_{i=1}^9 C_i = 167960 - 6917 = 161043$ (with $\sum_{i=1}^9 C_i = 1 + 2 + 5 + 14 + 42 + 132 + 429 + 1430 + 4862 = 6917$).

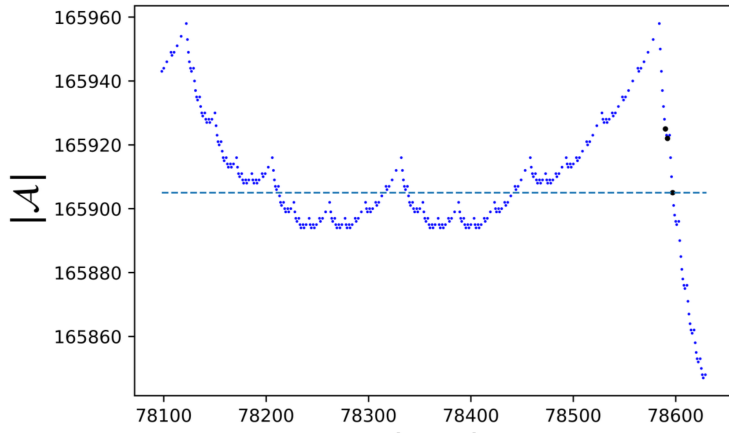
(11, 10)FSFA size zooming in.



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(11, 10)FSFA size zooming in more.

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2.PNG

3. Triangular sequences: Applying Binomials and discovering Catalan Numbers

A sequence $S(t'_i)$: $t'_4, t'_3, t'_2, t'_1, t'_0$

3 2 1 0 -1
 2 1 0 -1
 1 0 -1
 0 -1
 -1;
 2 1 0 -1
 1 0 -1
 0 -1
 -1;
 1 0 -1
 0 -1
 -1;
 0 -1
 -1;
 -1.

- (1) For each triangle t'_i and for the sequence $S(t'_i)$ find the number of terms, row sums, triangle sums, graph the terms and partial sums, maximum value.
- (2) Try the same questions with the sequence S_4^7 : $t'_4, t'_3, t'_2, t'_1, t'_0; t'_3, t'_2, t'_1, t'_0; t'_2, t'_1, t'_0; t'_1, t'_0; t'_0$.

Extra challenges:

A sequence $S(t_i)$: t_4, t_3, t_2, t_1, t_0

4 3 2 1 0
 3 2 1 0
 2 1 0
 1 0
 0;
 3 2 1 0
 2 1 0
 1 0
 0;
 2 1 0
 1 0
 0;
 1 0
 0;
 0.

A sequence of powers of 2 such as U_4, U_3, U_2, U_1, U_0 with $U_4 =$
 16,8,4,2,1
 8,4,2,1
 4,2,1
 2,1
 1

Acknowledgements

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