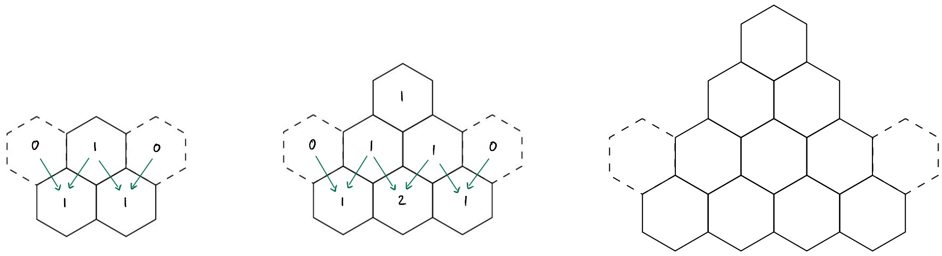
ARROW-CHASING WORLD

Activity 1 (Construction of Pascal's Triangle).

1. Consider the three figures below. Start by placing a 1 at the top center. The next row is constructed by adding adjacent elements in the previous row. Because there is nothing next to the 1 in the top row, the adjacent elements are considered to be 0. This process is repeated to produce each subsequent row.

   Fill the rightmost pattern with numbers.



2. **Notation of Pascal's Triangle:** The topmost row in Pascal's triangle is considered to be the 0-th row. The next row down is the 1-st row, then the 2-nd row, and so on. The leftmost element in each row is considered to be the 0-th element in that row. Then, to the right of that element is the 1-st element in that row, then the 2-nd element in that row, and so on (see Figure 1).

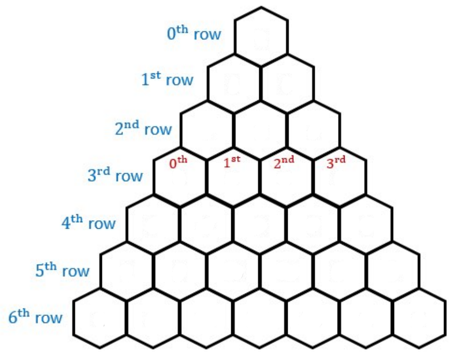


Figure 1: Notation of Pascal's triangle.

We denote the k-th entry in the n-th row of Pascal's Triangle by $\binom{n}{k}$ (read "n choose k"). Thus, the first entries of Pascal's Triangle are $\binom{0}{0} = 1$, $\binom{1}{0} = 1$, $\binom{1}{1} = 1$, $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, and so on. We call the numbers $\binom{n}{k}$ binomial coefficients. We have $\binom{n}{0} = \binom{n}{n} = 1$ for all n and

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq n$ by construction of Pascal's triangle.

(a) Fill Figure 1 with numbers.

(b) State the value of the following binomial coefficients: $\binom{5}{0}, \binom{4}{2}, \binom{3}{1}, \binom{5}{3}, \binom{2}{2}$.

(c) Explain why the formula $\binom{n}{k} = \binom{n}{n-k}$ holds true for $0 \leq k \leq n$.

(d) The 11-th row of Pascal's triangle contains the binomial coefficients of the form $\binom{10}{k}$ for $0 \leq k \leq 10$ and reads

$$1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1.$$

Use this to determine the values of the binomial coefficients of the form $\binom{11}{k}$ for $0 \leq k \leq 11$.

Activity 2.

1. We add all entries in a row of Pascal's triangle. Complete the table

Row	0	1	2	3	4	5	6
Sum of entries	1	2	4				

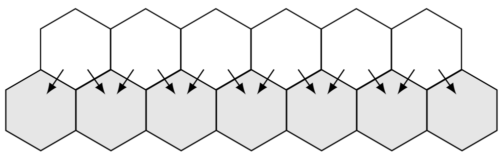
2. The sum of all entries of the 0-th row of Pascal's triangle equals 1. The sum of all entries of the 1-st row of Pascal's triangle equals 2. Conjecture a formula:

   The sum of all entries of the n-th row of Pascal's triangle equals $\boxed{}$.

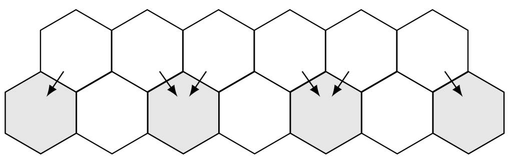
   Written in symbols:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \boxed{}. \qquad (1)$$

3. The figure below shows the $(n-1)$-th row an the n-th row of Pascal's triangle. Use it to explain why (1) holds true.

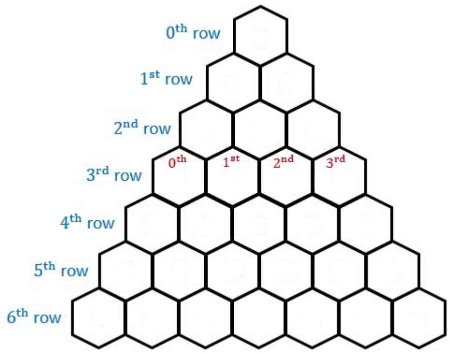


4. We add every other entry – i.e., the 0-th, the 2-nd, and so on – of a row of Pascal's triangle. Consider examples, make a conjecture and substantiate your conjecture using the following figure.



Activity 3 (Hockey Stick or Christmas Stocking?).

1. Compute $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}$ and express your result as a binomial coefficient. Mark the entries corresponding to each of the summands and the result in Pascal's triangle below.



2. Consider other sums of the same form and express the results as binomial coefficients. Make a conjecture for a general formula.

3. Ask for a hint sheet with a figure and use it to substantiate your claim.

4. Compute $\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3}$ and express your result as a binomial coefficient. Mark the entries corresponding to each of the summands and the result in Pascal's triangle. Consider other examples with the same pattern, make a conjecture for a general formula and substantiate your claim using an "arrow-chasing figure".

Activity 4 (Alternating row sums).

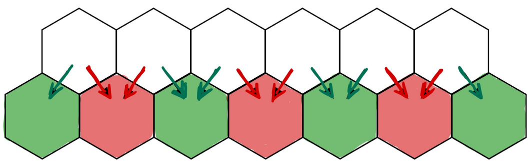
1. Compute $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$.

2. Compute similar expressions of the form

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = +\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots \pm \binom{n}{n}$$

and make a conjecture.

3. Substantiate your claim by using the figure below.



Activity 5 (Multi-arrows). We consider sums of the form

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n}.$$

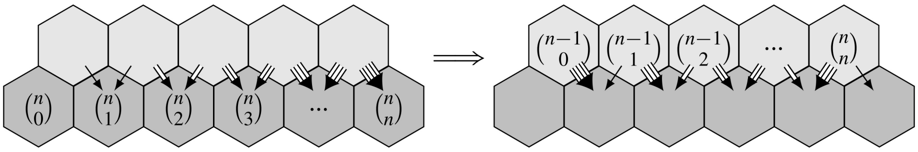
1. Complete the table.

n	0	1	2	3	4	5	6
$\sum_{k=0}^{n} k \cdot \binom{n}{k}$	0	1	4				

2. Make a conjecture for a closed formula

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = \boxed{}.$$

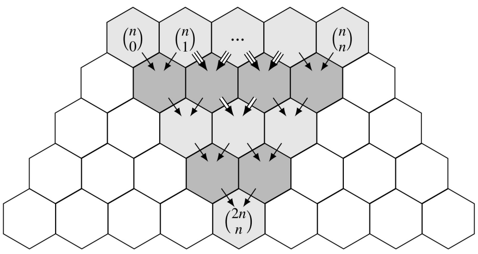
3. Substantiate your conjecture by using the figure below.



Activity 6 (Lagrange's Identity).
Prove the so-called Lagrange's Identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

by using the figures below.



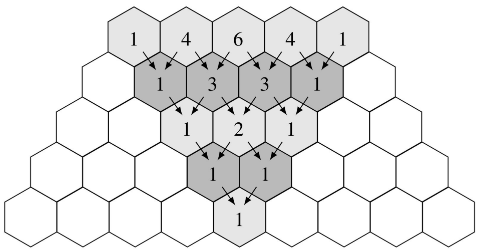


Figure 1 is taken (and adapted) from `https://brilliant.org/wiki/pascals-triangle/`. All other figures and the main ideas for this World are taken from the wonderful article
Krapf, R.: *Arrow-chasing in Pascal's triangle – Visual proofs for summation formulas involving binomial coefficients.* `https://doi.org/10.48550/arXiv.2508.16388`