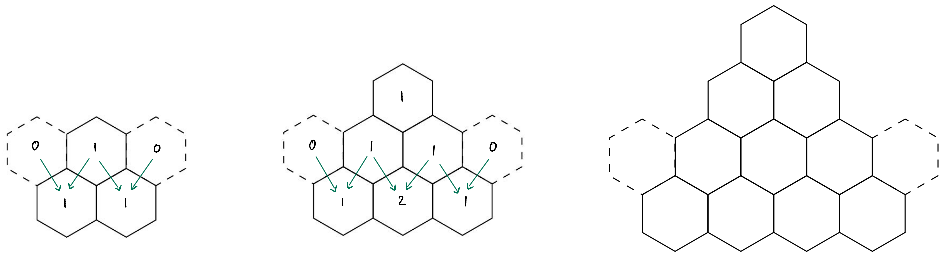
Arrow-Chasing World

Teacher's Version

Activity 1 (Construction of Pascal's Triangle).

1. Consider the three figures below. Start by placing a 1 at the top center. The next row is constructed by adding adjacent elements in the previous row. Because there is nothing next to the 1 in the top row, the adjacent elements are considered to be 0. This process is repeated to produce each subsequent row.

   Fill the rightmost pattern with numbers.



2. **Notation of Pascal's Triangle:** The topmost row in Pascal's triangle is considered to be the 0-th row. The next row down is the 1-st row, then the 2-nd row, and so on. The leftmost element in each row is considered to be the 0-th element in that row. Then, to the right of that element is the 1-st element in that row, then the 2-nd element in that row, and so on (see Figure 1).

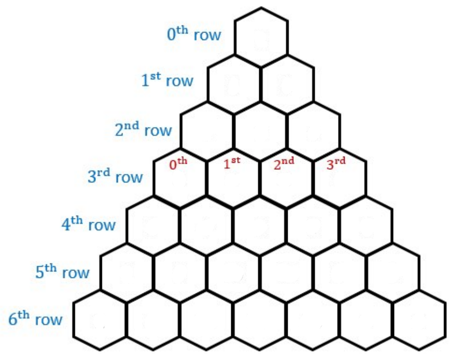


Figure 1: Notation of Pascal's triangle.

We denote the k-th entry in the n-th row of Pascal's Triangle by $\binom{n}{k}$ (read "n choose k"). Thus, the first entries of Pascal's Triangle are $\binom{0}{0} = 1$, $\binom{1}{0} = 1$, $\binom{1}{1} = 1$, $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, and so on. We call the numbers $\binom{n}{k}$ binomial coefficients. We have $\binom{n}{0} = \binom{n}{n} = 1$ for all n and

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq n$ by construction of Pascal's triangle.

(a) Fill Figure 1 with numbers.

(b) State the value of the following binomial coefficients: $\binom{5}{0}, \binom{4}{2}, \binom{3}{1}, \binom{5}{3}, \binom{2}{2}$.

(c) Explain why the formula $\binom{n}{k} = \binom{n}{n-k}$ holds true for $0 \leq k \leq n$.

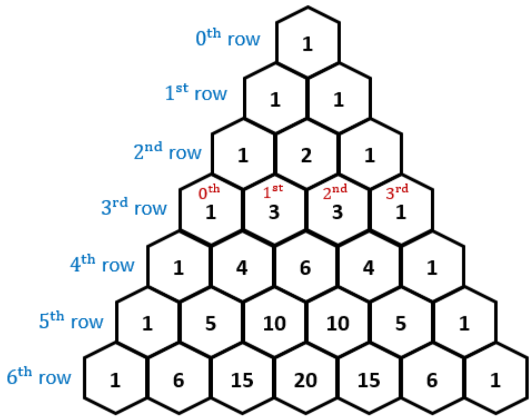
(d) The 11-th row of Pascal's triangle contains the binomial coefficients of the form $\binom{10}{k}$ for $0 \leq k \leq 10$ and reads

$$1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1.$$

Use this to determine the values of the binomial coefficients of the form $\binom{11}{k}$ for $0 \leq k \leq 11$.

Solutions to Activity 1.

(a)



(b) $\binom{5}{0} = 1$, $\binom{4}{2} = 6$, $\binom{3}{1} = 3$, $\binom{5}{3} = 10$, $\binom{2}{2} = 1$.

(c) The triangle ist symmetric by definition.

(d) The 12-th row of Pascal's triangle conatains the binomial coefficients of the form $\binom{11}{k}$ for $k = 0, \ldots, 11$ and reads

$$1 \quad 11 \quad 55 \quad 165 \quad 330 \quad 562 \quad 462 \quad 330 \quad 165 \quad 55 \quad 11 \quad 1.$$

Activity 2.

1. We add all entries in a row of Pascal's triangle. Complete the table

Row	0	1	2	3	4	5	6
Sum of entries	1	2	4				

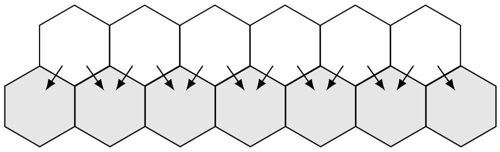
2. The sum of all entries of the 0-th row of Pascal's triangle equals 1. The sum of all entries of the 1-st row of Pascal's triangle equals 2. Conjecture a formula:

   The sum of all entries of the n-th row of Pascal's triangle equals $\boxed{}$.

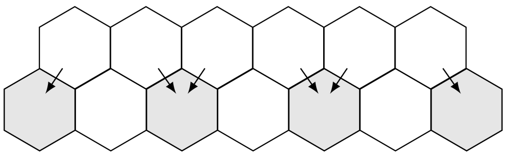
   Written in symbols:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \boxed{}. \qquad (1)$$

3. The figure below shows the $(n-1)$-th row an the n-th row of Pascal's triangle. Use it to explain why (1) holds true.



4. We add every other entry – i.e., the 0-th, the 2-nd, and so on – of a row of Pascal's triangle. Consider examples, make a conjecture and substantiate your conjecture using the following figure.



Solutions to Activity 2.

1.

Row	0	1	2	3	4	5	6
Sum of entries	1	2	4	8	16	32	64

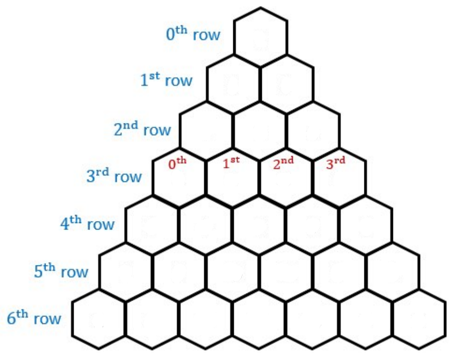
2. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

3. The figure shows that each entry of row $n-1$ is added twice to obtain row n. Thus, the sum of all entries in row n is twice as big as the sum of all entries in row $n-1$. As the sum of the entries of row 0 equals 1, the proof is complete.

4. $\displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = 2^{n-1}$. The figure shows that each entry of row $n-1$ is added exactly once to obtain row n. Thus, the sum of every other entry in row n equals the sum of all entries of row $n-1$, which is 2^{n-1}.

Activity 3 (Hockey Stick or Christmas Stocking?).

1. Compute $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}$ and express your result as a binomial coefficient. Mark the entries corresponding to each of the summands and the result in Pascal's triangle below.



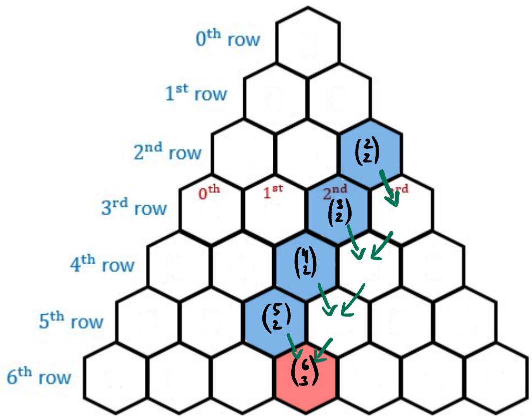
2. Consider other sums of the same form and express the results as binomial coefficients. Make a conjecture for a general formula.

3. Ask for a hint sheet with a figure and use it to substantiate your claim.

4. Compute $\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3}$ and express your result as a binomial coefficient. Mark the entries corresponding to each of the summands and the result in Pascal's triangle. Consider other examples with the same pattern, make a conjecture for a general formula and substantiate your claim using an "arrow-chasing figure".

Solutions to Activity 3.

1. $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}=\binom{6}{3}$.



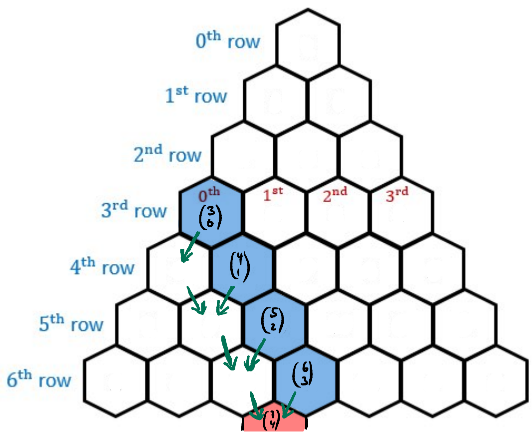
2. We conjecture $\displaystyle\sum_{m=k}^{n}\binom{m}{k}=\binom{n+1}{k+1}$ or $\displaystyle\sum_{r=0}^{\ell}\binom{k+r}{k}=\binom{k+\ell+1}{k+1}$.

3. The figure above shows the proof (consider the arrows).

4. $\binom{3}{0}+\binom{4}{1}+\binom{5}{2}+\binom{6}{3}=\binom{7}{3}$. We conjecture

$$\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+r+1}{r}.$$

The marking and the arrow-chasing proof are in the figure below.



Alternatively, one can prove this identity by simply reflecting the picture at the symmetry line of Pascal's triangle, which gives the same picture as before.

Activity 4 (Alternating row sums).

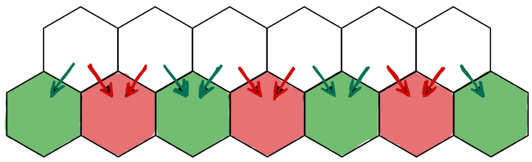
1. Compute $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$.

2. Compute similar expressions of the form

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = +\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots \pm \binom{n}{n}$$

and make a conjecture.

3. Substantiate your claim by using the figure below.



Solutions to Activity 4. We have $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$, which is proven by the figure as each entry from row $n-1$ is counted once positively and once negatively, so everything cancels out.

Activity 5 (Multi-arrows). We consider sums of the form

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n}.$$

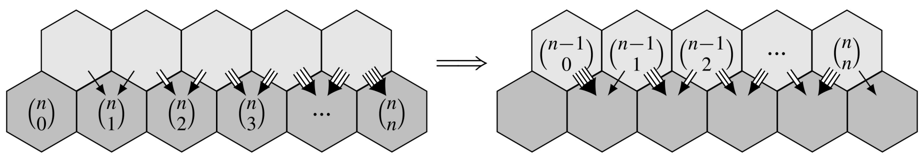
1. Complete the table.

n	0	1	2	3	4	5	6
$\sum_{k=0}^{n} k \cdot \binom{n}{k}$	0	1	4				

2. Make a conjecture for a closed formula

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = \boxed{}.$$

3. Substantiate your conjecture by using the figure below.



Solutions to Activity 5 (Multi-arrows).

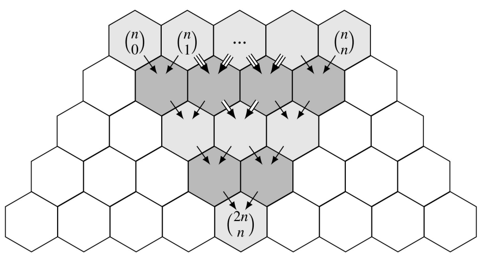
We have $\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$. The proof is given by the picture above by applying the symmetry of Pascal's triangle to see that each entry of row $n-1$ is counted n times.

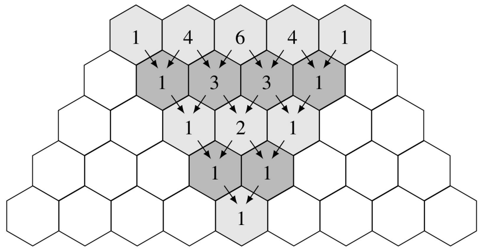
Activity 6 (Lagrange's Identity).

Prove the so-called Lagrange's Identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

by using the figures below.





Solutions to Activity 6 (Lagrange's Identity).

The binomial coefficient $\binom{2n}{n}$ is the middle entry in the $2n$-th row of Pascal's triangle. By successively using the defining property of the triangle we see that this entry is the sum of the entries above, where each entries value of row n tells us how often it is counted (you find an upside-down copy of Pascal's triangle, see the figure on the right-hand side).

Figure 1 is taken (and adapted) from `https://brilliant.org/wiki/pascals-triangle/`. All other figures and the main ideas for this World are taken from the wonderful article

Krapf, R.: *Arrow-chasing in Pascal's triangle – Visual proofs for summation formulas involving binomial coefficients.* `https://doi.org/10.48550/arXiv.2508.16388`