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Arrow-chasing in Pascal's triangle – Visual proofs for summation formulas involving binomial coefficients

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Binomial coefficients play an important role in combinatorics and in probability theory. The standard definition involves factorials:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

However, historically, binomial coefficients were first considered as entries in the so-called *Pascal triangle*. Pascal's triangle was known in many cultures, e.g., by the Persian mathematician Al-Karaji (953–1029), by the Chinese mathematician Yang Hui (1238–1298) and by the European mathematician Jordanus de Nemore (fl. 13th century). Binomial coefficients can thus be defined as numbers $\binom{n}{k}$ for $0 \leqslant k \leqslant n$ which satisfy the following recurrence relation, called *Pascal's rule*:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \text{ with } \binom{n}{0} = \binom{n}{n} = 1$$

In addition, we define $\binom{n}{k}$ to be 0 for $k > n$. It is a standard exercise to show that both definitions are equivalent and, moreover, that $\binom{n}{k}$ counts the number of ways to choose k out of n objects. Using Pascal's rule one obtains Pascal's triangle by placing 1s in the outermost entries of a number triangle, and then filling each inner entry with the sum of the two numbers diagonally above. By construction, Pascal's triangle is symmetrical.

Arrows

There are many known summation formulas involving binomial coefficients, the most prominent of which is probably the following:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

In Pascal's triangle, this means that the sum of the entries in the n-th row* is given by 2^n. Standard proofs of this identity use the binomial theorem, induction or combinatorial arguments. However, one can directly prove it using Pascal's triangle. To see this, we visualize Pascal's rule in Figure 1. In order to obtain the sum of all entries in the n-th row, we add each entry in the row above twice (see Figure 2). Hence, the row sum is twice the previous row sum. Since the row sum in row 0 is $1 = 2^0$, this proves the claim.

*Since we usually start with $n = 0$, we denote by the n-th row the row which contains all binomial coefficients of the form $\binom{n}{k}$.

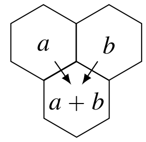

Figure 1 Pascal's rule

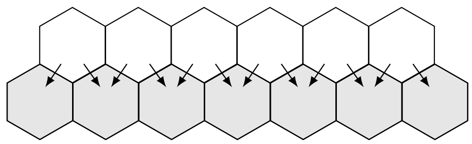

Figure 2 Row sum in Pascal's triangle

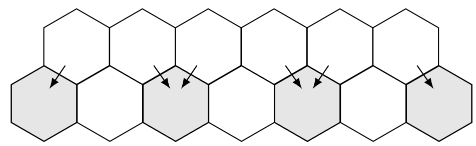

Figure 3 Sum of half of the entries in a row

Almost the same proof can be used to prove that if we only add every second entry in the n-th row of Pascal's triangle we obtain exactly the sum of all entries in the row above, i.e., 2^{n-1} (see Figure 3).

While these particular proofs are not new (they have previously been published by Ángel Plaza in [**1**] and [**2**]), the method of drawing arrows in Pascal's triangle – which we denote by *arrow-chasing* – offers a powerful tool for discovering a wide range of new proofs of identities involving binomial coefficients. A very useful identity is the following:

Theorem 1 (Hockey stick identity)**.** *For all $n, k \in \mathbb{N}$ such that $k \leqslant n$ we have:*

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$

Proof. As displayed in Figure 4, we can interpret $\sum_{m=k}^{n} \binom{m}{k}$ as the sum of the entries in the k-th diagonal from the k-th to the n-th row (marked light gray). If we add these entries by successively using Pascal's rule and the fact that $\binom{k}{k} = 1 = \binom{k+1}{k+1}$, we obtain $\binom{n+1}{k+1}$ (marked dark gray). ■

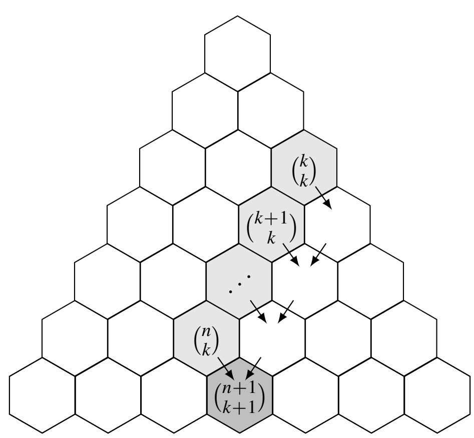

Figure 4 Proof of the hockey stick identity

When searching for the hockey stick identity in the standard literature, one usually finds the visualization (without arrows) in the shape of a hockey stick in Pascal's triangle. However, the formula is still proven – despite being directly derived from the illustration by drawing arrows – using either induction or a combinatorial proof.

Multi-arrows

The strategy of proving summation formulas using arrows in Pascal's triangle can be generalized to the use of so-called multi-arrows. In this case, instead of a single arrow, multiple arrows (referred to as a *multi-arrow*) are used. The arity of a multi-arrow (the number of arrows it comprises) indicates the coefficient, or *weight*, by which the corresponding entry is multiplied. An easy example that illustrates this technique is the following:

Theorem 2. *For every $n \in \mathbb{N}$ we have:*

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$$

Proof. We view $k \cdot \binom{n}{k}$ as an entry in the n-th row of Pascal's triangle, multiplied by k (see Figure 5). Since each entry in Pascal's triangle is the sum of the two entries located diagonally above it, we interpret this as a k-fold arrow pointing from the entries above towards the corresponding entry. Now, using the symmetry of Pascal's triangle, we rearrange these arrows so that from each entry in the $(n-1)$-th row, exactly n arrows originate. Hence, the sum $\sum_{k=0}^{n} k \cdot \binom{n}{k}$ is equal n times the sum of all entries in the $(n-1)$-th row, i.e., $n \cdot 2^{n-1}$. ∎

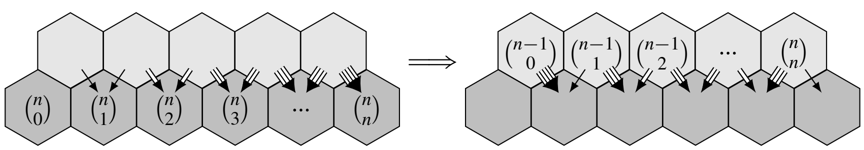

Figure 5 Proof of Theorem 2

Theorem 3 (Lagrange's identity)**.** *For every $n \in \mathbb{N}$ we have:*

$$\sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Proof. We interpret $\binom{2n}{n}$ as the middle entry in the $(2n)$-th row of Pascal's triangle (see Figure 6). Note that $\binom{2n}{n}$ is the result of summing the two entries located diagonally above it. These, in turn, are also obtained by adding the entries above. However, the middle entry in row $2n-2$, i.e., $\binom{2n-2}{n-1}$ is needed twice, so we have double arrows pointing towards it. This process can be continued in such a way that the arity each of the arrows pointing to each field, from both the left and right, equals the sum of the arity of the arrows originating from that field. This ensures that the weighted sum of the entries in a row – where each entry is multiplied by the arity of the arrows pointing towards it – remains equal to $\binom{2n}{n}$. Observe that, as illustrated below, the arities of the arrows form an inverted version of Pascal's triangle. Therefore, at the n-th row, the number of arrows originating from each entry matches the value of the entry itself. Hence, $\binom{2n}{n}$ is equal to the weighted sum of the entries in the n-th row which is $\sum_{k=0}^{n} \binom{n}{k}^2$. ∎

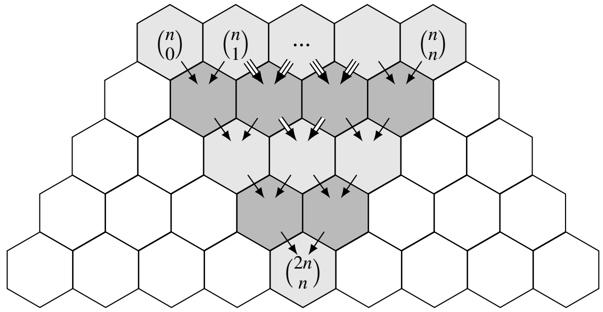

Figure 6 Proof of Lagrange's identity

For large n, drawing multi-arrows of high arity becomes impractical. Instead, we place a single arrow and simply write the arity in each corresponding field. The weighted sum of the binomial coefficients, each multiplied by the arity in its field, in a given row remains constant and equal to $\binom{2n}{n}$. An example of this notation for $n = 4$ is shown in Figure 7.

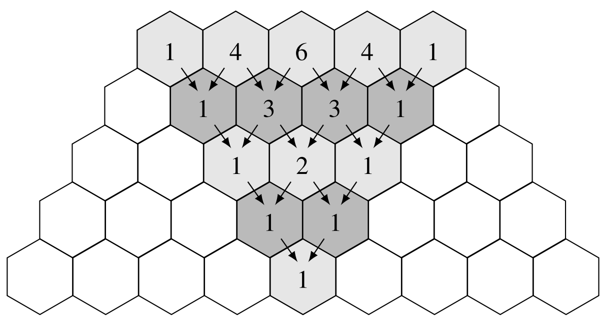

Figure 7 Simplified proof of Lagrange's identity

Essentially the same proof can be used to derive the more general formula called *Chu-Vandermonde identity*. This identity is often named only after the French mathematician Alexandre-Théophile Vandermonde (who published its proof in 1772), although it was already known in 1303 by the Chinese mathematician Zhu Shijie (see [**3**]).

Theorem 4 (Chu-Vandermonde identity)**.** *For $m, n, k \in \mathbb{N}$ such that $k \leqslant m, n$ we have:*

$$\sum_{j=0}^{k} \binom{m}{j} \cdot \binom{n}{k-j} = \binom{m+n}{k}$$

We will leave this as an exercise to the reader. Instead, we prove an identity that can be found in [**4**, p. 148], which Grinberg [**5**] refers to as the *upside-down Chu-Vandermonde identity*.

Theorem 5 (Upside-down Chu-Vandermonde identity)**.** *Let $n, k, l \in \mathbb{N}$ be natural numbers such that $k + l \leqslant n$. Then we have the following identity:*

$$\sum_{j=k}^{n-l} \binom{j}{k} \binom{n-j}{l} = \binom{n+1}{k+l+1}$$

Proof. We view $\binom{j}{k}\binom{n-j}{l}$ als entries of the k-th diagonal of Pascal's triangle with weights given by $\binom{n-j}{l}$ (which are the entries of the l-th diagonal in reverse order), as depicted in Figure 8 for $n = 7, k = 1$ and $l = 3$. If we add these entries as shown below, we obtain a parallelogram, whose lowest vertex is $\binom{n+1}{k+l+1}$ (marked dark gray). Hence, the sum is equal to $\binom{n+1}{k+l+1}$.

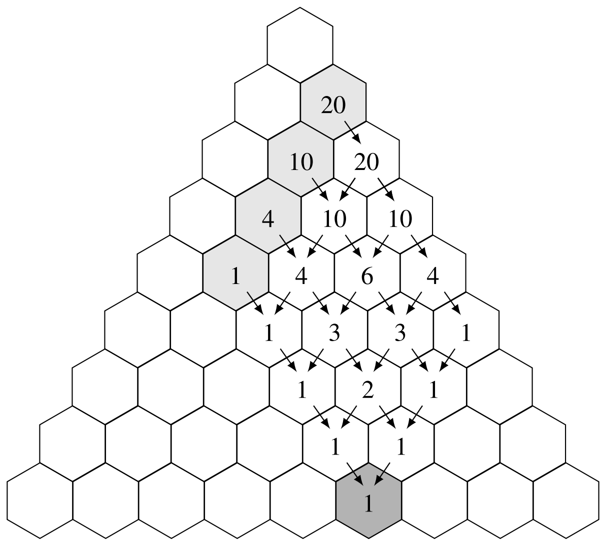

Figure 8 Proof of Theorem 5

■

As a last application of multi-arrows and our simplified notation, we consider two variations of Theorem 2:

Theorem 6. *For each $k \in \mathbb{N}$ we have:*

$$\sum_{k\geqslant 0} k \cdot \binom{n}{2k} = n \cdot 2^{n-3} \quad \textit{and} \quad \sum_{k\geqslant 0} k \cdot \binom{n}{2k+1} = (n-2) \cdot 2^{n-3}$$

Proof. We consider the n-th row of Pascal's triangle, where we add only the weighted even-indexed or the weighted odd-indexed entries (see Figure 9).

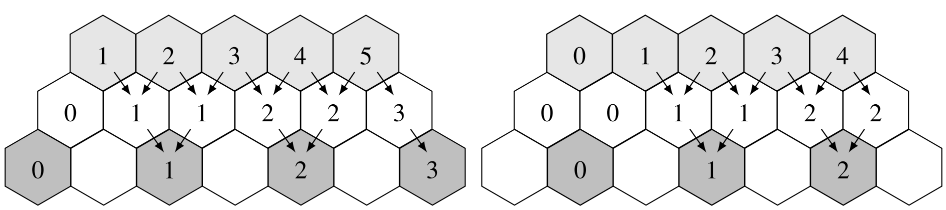

Figure 9 Proof of Theorem 6

Using arrow-chasing, we move to row $n - 2$. The second identity then clearly follows from Theorem 2 for $n - 2$ since we obtain weight k in the k-th entry of the $(n - 2)$-th row. The weighted sum of the even-indexed entries of the n-th row is similar, but instead we obtain weight $k+1$ in the k-th entry of the $(n-2)$-th row. Hence, we obtain the same sum as in the second identity but we additionally have to add the row sum in the $(n-2)$-th row, which is 2^{n-2}. Therefore, we obtain $(n-2)\cdot 2^{n-3} + 2^{n-2} = n \cdot 2^{n-3}$. ∎

Arrows with negative arity

Arrow-chasing can also be applied to prove identities involving alternating sums of binomial coefficients. To accommodate this, we will now allow arrows to have negative arity.

Theorem 7. *For every $n \in \mathbb{N}$ with $n > 0$ we have:*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof. Each entry in the $(n-1)$-th row is both added and subtracted once (see Figure 10). This means that each entry in that row has weight 0 and hence the weighted row sum is also equal to 0. Since by construction the weighted sum in the n-th row is equal to that of the $(n-1)$-th row, the claim follows. ∎

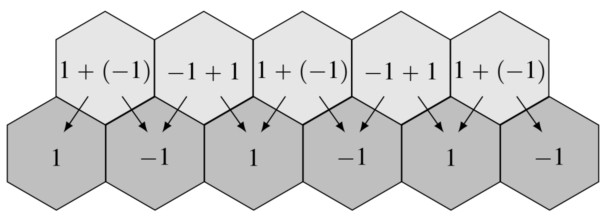

Figure 10 Proof of Theorem 7

Essentially the same proof works also for the formula

$$\sum_{k=0}^{n} (-1)^k k \binom{n}{k} = 0.$$

Note that as in the case of positive weights, one can simply add the weights in order to obtain those of the row located above (see Figure 11). The identity then follows directly from Theorem 7.

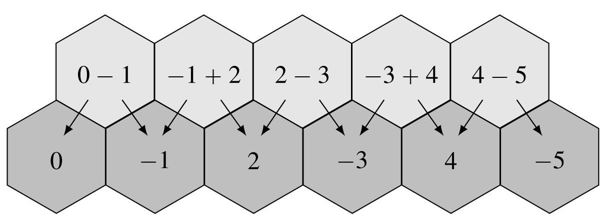

Figure 11 Alternating weighted row sum

If we generalize these two statements further, we obtain the following:

Theorem 8. *For natural numbers $m < n$ we have:*

$$\sum_{k=m}^{n} (-1)^k \binom{n}{k}\binom{k}{m} = 0$$

Displayed below is the case $n = 7$ and $m = 2$.

Proof. We consider the n-th row of Pascal's triangle, starting at the entry $\binom{n}{m}$ (see Figure 12). Now, the weight of each binomial coefficient $\binom{n}{k}$ is given by $(-1)^k\binom{k}{m}$. Thus, the weights correspond to the alternating entries of the m-th diagonal. Hence, the differences in the row above correspond to the alternating entries of the $(m-1)$-th diagonal, and so on. In the $(n-m)$-th row, the entries are therefore given by the alternating entries of the 0-th diagonal, which are ± 1. Thus, the weighted sum is the alternating sum of the entries of the $(n-m)$-th row, which is 0. ■

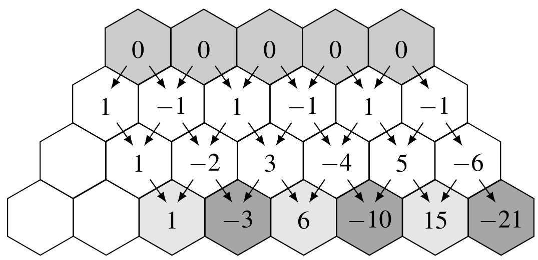

Figure 12 Proof of Theorem 8

Using the same proof idea, we can generalize Theorem 12 further in order to obtain the following identity:

$$\sum_{k=m}^{n} (-1)^{m-k} \binom{n}{k}\binom{k+p-m}{p} = \binom{n-p-1}{n-m}$$

Next, we prove a variant of the hockey stick identity, which was shown in [**6**]. However, our proof is a lot shorter than the one presented in [**6**].

Theorem 9. *For all $n, k \in \mathbb{N}$ such that $k \leqslant n$ we have:*

$$\sum_{m=k}^{n} m\binom{m}{k} = n\binom{n+1}{k+1} - \binom{n+1}{k+2}$$

This identity does not involve an alternating sum, but we still need negative arities due to the negative sign in front of $\binom{n+1}{k+2}$.

Proof. As illustrated in Figure 13, we begin with the entries $\binom{n+1}{k+1}$ (with weight n) and $\binom{n+1}{k+2}$ (with weight -1). Using arrow-chasing, we can determine the weights of the entries in the k-th, $(k+1)$-th and $(k+2)$-th diagonals, as displayed below for $k = 2$. Clearly, the weight of $\binom{m}{k}$ is given by m, which establishes the claim. ■

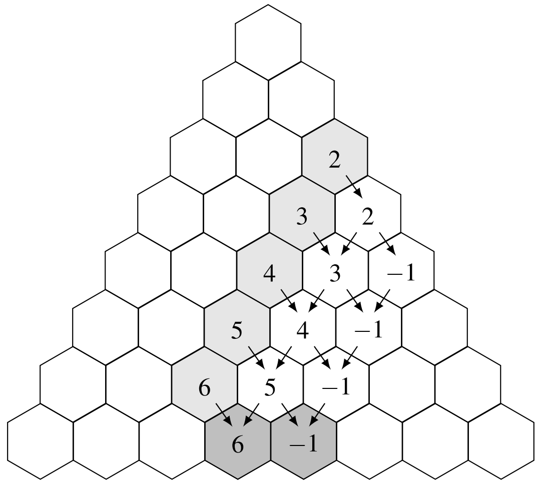

Figure 13 Proof of Theorem 9

Real-valued and complex-valued arrows

We can further generalize our method by allowing arrows to have a real-valued or even complex-valued arity. Our first example features arrows of rational arity.

Theorem 10. *For every $n \in \mathbb{N}$ we have:*

$$\sum_{k=0}^{n} \binom{n+k}{2k} 2^{-k} + \sum_{k=0}^{n-1} \binom{n+k}{2k+1} 2^{-(k+1)} = 2^n$$

Proof. Observe that both sums can be viewed as two subsequent weighted diagonal sums in Pascal's triangle (see Figure 14). Using Pascal's rule as well as $\frac{1}{2^k} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}}$, we can see that the left-hand side is equal to $\frac{1}{2^n}$ times the row sum in the $(2n)$-th row (marked light gray). Now, since the row sum in the $(2n)$-th row of Pascal's triangle is always equal to 2^{2n}, we obtain $\frac{1}{2^n} \cdot 2^{2n} = 2^n$, as desired. ■

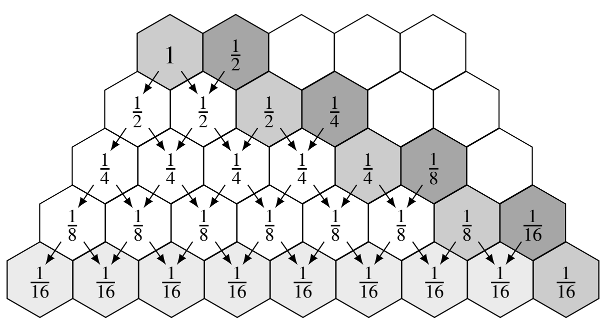

Figure 14 Proof of Theorem 10

The following identity, denoted *Boscarol's identity*, was first proved in 1982 (see [**7**]).

Theorem 11 (Boscarol's identity)**.** *For all $m, n \in \mathbb{N}$ we have:*

$$\sum_{k=0}^{m} \frac{\binom{n+k}{k}}{2^{n+k}} + \sum_{k=0}^{n} \frac{\binom{n+m-k}{m}}{2^{n+m-k}} = 2$$

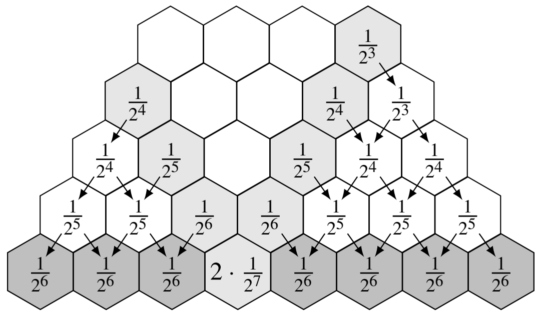

Figure 15 Proof of Boscarol's identity for $n = 4$ and $m = 3$

Proof. We consider the diagonal starting at $\binom{n}{0}$ going to the right as well as the diagonal starting at $\binom{m}{m}$ going to the left, equpipped with weights $\frac{1}{2^{n+k}}$ and $\frac{1}{2^{n+m-k}}$, respectively (see Figure 15). These diagonals intersect at $\binom{n+m}{m}$, which is counted twice and thus obtains weight $2 \cdot \frac{1}{2^{n+m}} = \frac{1}{2^{n+m-1}}$. Now, as in the proof of Theorem 10, we can move down to the $(n+m)$-th row, where we obtain weight $\frac{1}{2^{n+m-1}}$ in each entry. Hence, we obtain $\frac{1}{2^{n+m-1}} \cdot 2^{n+m} = 2$. ■

The most prominent application of complex-valued arrows is the binomial theorem. In this case, the arrow-chasing argument is well-known:

Theorem 12 (Binomial Theorem)**.** *For all complex numbers a and b we have:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. We proceed by induction. For $n = 0$, this is obvious. Now, let us assume that the claim holds for some natural number n. Hence, we can interpret $(a+b)^n$ as a weighted row sum with weights a^{n-k} and b^k. In order to obtain $(a+b)^{n+1}$, we multiply each weight with $(a+b)$, as illustrated in Figure 16. ■

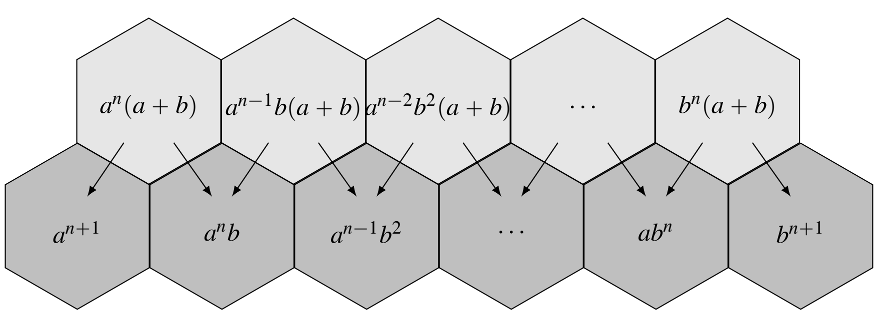

Figure 16 Proof of the binomial theorem

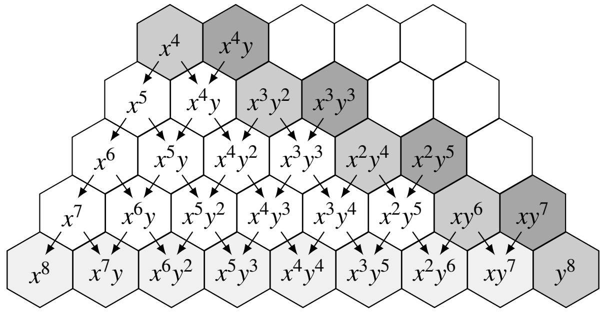

Figure 17 Proof of Theorem 13 for $n = 4$

The binomial theorem is also useful for proving other binomial identities via arrow-chasing. For example, we can generalize Theorem 10 in order to obtain the following identity:

Theorem 13. *For all $n \in \mathbb{N}$ and for all $x, y \in \mathbb{R}$ such that $x + y = 1$ we have:*

$$\sum_{k=0}^{n} \binom{n+k}{2k} x^{n-k} y^{2k} + \sum_{k=0}^{n-1} \binom{n+k}{2k+1} x^{n-k} y^{2k+1} = 1$$

Proof. The proof of Theorem 13 (see Figure 17) is a direct generalization of the proof of Theorem 10 using $x^{k+1}y^l + x^k y^{l+1} = (x+y)x^k y^l = x^k y^l$ and the binomial theorem in the $(2n)$-th row of Pascal's triangle:

$$\sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k} y^k = (x+y)^{2n} = 1$$

■

Similarly, one can generalize Boscarol's identity in the following way:

Theorem 14 (Gosper's identity). *For all $m, n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ such that $x + y = 1$ we have:*

$$\sum_{k=0}^{m} \binom{n+k}{k} x^{n+1} y^k + \sum_{k=0}^{n} \binom{m+k}{k} x^k y^{m+1} = 1$$

Clearly, this identity generalizes Boscarol's identity (take $x = y = \frac{1}{2}$ and observe that $\binom{m+k}{k} = \binom{n+m-j}{m}$ for $j = n - k$). Additionally, it extends Gosper's original identity, which corresponds to the special case $m = n$ [**8**]. Several interesting proofs exist for this identity, including a probabilistic proof [**9**] or – for a special case – a proof using differential equations [**10**]. The proof of Theorem 14 via arrow-chasing uses the same idea as the one of Theorem 13 is left as an exercise to the reader.

Horizontal arrows

Sometimes, it is useful to reformulate Pascal's rule as a subtraction formula. This gives us horizontal arrows, where each entry is calculated as the difference between the entry diagonally below to the right and the adjecent entry directly to its right.

We use this approach in order to prove the following summation formula:

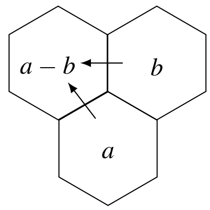

Figure 18 Horizontal interpretation of Pascal's rule

Theorem 15. *For each $n \in \mathbb{N}$ we have:*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n} = 1$$

For the proof, consider the following diagram (see Figure 19), where we draw multi-arrows so that the idea becomes clearer:

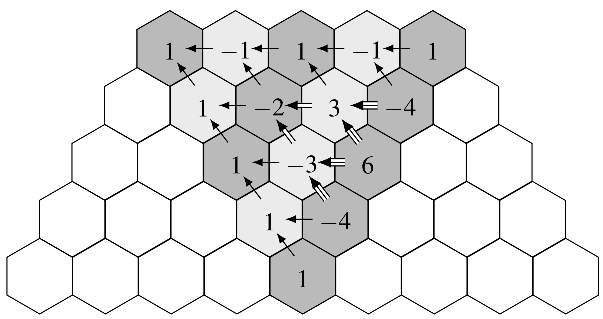

Figure 19 Proof of Theorem 15

Proof. We interpret the desired result 1 as $1 = \binom{n}{0}$, i.e., as the first entry in the n-th row of Pascal's triangle. Now, as observed above, we can write this as the difference of the entry diagonally below to the right and the neighboring entry to the right. We repeat this procedure diagonal by diagonal and compute the corresponding weight of each entry. For example, for $n = 4$, as shown above, $\binom{5}{2}$ obtains the weight -2, since we need to subtract $\binom{5}{2}$ to obtain $\binom{5}{1}$ and we need to add $-\binom{5}{2}$ in order to obtain $-\binom{4}{2}$.

This procedure has the effect that the weighted diagonal sum always remains constant and hence equal to 1. Moreover, by construction, the weights provide an alternating rotated version of Pascal's triangle. The final diagonal has the entries $\binom{2n-k}{n}$ and its weights correspond exactly to the alternating entries of the n-th row, i.e., $(-1)^k\binom{n}{k}$. ■

By shifting the rotated alternating version of Pascal's triangle in Figure 19, this proof can easily be generalized in order to obtain the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{p+k}{m} = (-1)^n \cdot \binom{p}{m-n},$$

which was formulated by Knuth [**11**] and recently proven combinatorially in [**12**].

Conclusion

In this paper, we introduced a novel technique for constructing proofs via arrow-chasing in Pascal's triangle. This approach offers a variety of new visual proofs for many well-known summation formulas involving binomial coefficients. Since the technique relies solely on Pascal's rule, our arguments can be converted into algebraic proofs. However, one significant advantage of this method is its accessibility: it requires only basic arithmetic, making it approachable for a wide audience. Moreover, unlike many algebraic proofs, arrow-chasing not only proves an identity but also provides intuitive insight into *why* the identity holds. While combinatorial proofs (such as those presented in [**13**]) also have this explanatory aspect, our method is often simpler, especially when dealing with alternating sums.

We showed a few selected examples of arrow-chasing in this paper, but the technique has plenty of other applications. For example, it can also be used in related number arrays such as the Bell triangle or the Catalan triangle. We plan to explore these possibilities in a subsequent article.

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Summary This article demonstrates, using numerous examples of varying complexity, how one can visually prove summation formulas involving binomial coefficients by exclusively using the recurrence relation for binomial coefficients and its illustration through arrows in Pascal's triangle. The method developed for this purpose, which we call 'arrow-chasing', is elementary and it is accessible to a very broad audience.